

A VOLUME ESTIMATE FOR THE SET OF STABLE LATTICES

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ABSTRACT. We show that in high dimensions the set of stable lattices is almost of full measure in the space of unimodular lattices.

Let $G \stackrel{\text{def}}{=} \text{SL}_n(\mathbb{R})$, $\Gamma \stackrel{\text{def}}{=} \text{SL}_n(\mathbb{Z})$, and let $A \subset G$ denote the subgroup of diagonal matrices with positive entries. The quotient space $\mathcal{L}_n \stackrel{\text{def}}{=} G/\Gamma$ is naturally identified with the space of unimodular lattices in \mathbb{R}^n , and the group G (and any of its subgroups) acts via left translations, or equivalently, by acting on lattices via its linear action on \mathbb{R}^n . A lattice Λ is called *stable* if for any subgroup $\Delta \subset \Lambda$, one has $\text{vol}(\Delta \otimes \mathbb{R}/\Delta) \geq 1$ (in the literature the term *semi-stable* is also used), and we denote the set of stable lattices by $\mathcal{S}^{(n)}$.

A central problem is to understand the orbits of the A -action on \mathcal{L}_n . In [SW] we proved that for any lattice $\Lambda \in \mathcal{L}_n$, the orbit-closure $\overline{A\Lambda}$ contains a stable lattice. This result reduces the proof of Minkowski's conjecture on the product of inhomogeneous linear forms to that of estimating the Euclidean covering radius of stable lattices (see [SW] for details). Understanding stable lattices is therefore a natural problem due to its connection both with well-studied problems in the geometry of numbers, and with dynamics of the A -action. Although $\mathcal{S}^{(n)}$ is compact (while \mathcal{L}_n is not), in this note we show that $\mathcal{S}^{(n)}$ has almost full measure with respect to the natural probability measure on \mathcal{L}_n , for large n . Moreover the convergence to full measure is very fast. This answers a question we were asked by G. Harder, and can be viewed as a manifestation of the concentration of mass along the equator in high dimensional Euclidean balls.

We will prove the following.

Theorem 1. *Let m denote the G -invariant probability measure on \mathcal{L}_n derived from Haar measure on G , and let $\mathcal{S}^{(n)}$ denote the subset of stable lattices in \mathcal{L}_n . Then there is a constant $C > 0$ such that for all sufficiently large n ,*

$$m\left(\mathcal{L}_n \setminus \mathcal{S}^{(n)}\right) \leq \left(\frac{C}{n}\right)^{\frac{n-1}{2}}.$$

In particular $m\left(\mathcal{S}^{(n)}\right) \rightarrow 1$ as $n \rightarrow \infty$.

For $\Lambda \in \mathcal{L}_n$ and a subgroup $\Delta \subset \Lambda$, we denote by $r(\Delta)$ its *rank* and by $|\Delta|$ its *covolume* in the Euclidean subspace $\Delta \otimes \mathbb{R} \subset \mathbb{R}^n$. For $k = 1, \dots, n-1$ let us denote $\mathcal{V}_k(\Lambda) \stackrel{\text{def}}{=} \left\{ |\Delta|^{1/k} : \Delta \subset \Lambda, r(\Delta) = k \right\}$ and $\alpha_k(\Lambda) = \min \mathcal{V}_k(\Lambda)$ so that Λ is stable if and only if $\alpha_k(\Lambda) \geq 1$ for $k = 1, \dots, n-1$. Let

$$\mathcal{S}_k^{(n)}(t) \stackrel{\text{def}}{=} \{x \in \mathcal{L}_n : \alpha_k(x) \geq t\}, \quad \mathcal{S}_k^{(n)} \stackrel{\text{def}}{=} \mathcal{S}_k^{(n)}(1).$$

With this notation $\mathcal{S}^{(n)} = \bigcap_{k=1}^{n-1} \mathcal{S}_k^{(n)}$. We will show:

Proposition 2. *There is $C > 0$ such that for all sufficiently large n , and all $k \in \{1, \dots, n-1\}$,*

$$m\left(\mathcal{L}_n \setminus \mathcal{S}_k^{(n)}\right) \leq \frac{1}{n} \left(\frac{C}{n}\right)^{\frac{k(n-k)}{2}}. \quad (1)$$

Proof of Theorem 1. For $n > C$, the largest value of $\left(\frac{C}{n}\right)^{\frac{k(n-k)}{2}}$ is attained when $k = 1$ and $k = n-1$. Therefore (1) implies

$$\begin{aligned} m\left(\mathcal{L}_n \setminus \mathcal{S}^{(n)}\right) &= m\left(\mathcal{L}_n \setminus \bigcap_{k=1}^{n-1} \mathcal{S}_k^{(n)}\right) = m\left(\bigcup_{k=1}^{n-1} \mathcal{L}_n \setminus \mathcal{S}_k^{(n)}\right) \\ &\leq \frac{n-2}{n} \left(\frac{C}{n}\right)^{\frac{n-1}{2}} \leq \left(\frac{C}{n}\right)^{\frac{n-1}{2}}. \end{aligned}$$

□

We will also show:

Proposition 3. *There is $C_1 > 0$ such that if we set*

$$t_k = t(n, k) \stackrel{\text{def}}{=} \left(\frac{n}{C_1}\right)^{\frac{n-k}{2n}}, \quad (2)$$

then

$$\max_{k=1, \dots, n-1} m\left(\mathcal{L}_n \setminus \mathcal{S}_k^{(n)}(t_k)\right) = o\left(\frac{1}{n}\right).$$

In particular, $m\left(\bigcap_{k=1}^{n-1} \mathcal{S}_k^{(n)}(t_k)\right) \rightarrow_{n \rightarrow \infty} 1$.

Remarks. 1. Let us define $\bar{\alpha}_{n,k} \stackrel{\text{def}}{=} \sup \{\alpha_k(\Lambda) : \Lambda \in \mathcal{L}_n\}$. These quantities are powers of the so-called *Rankin constants* or *generalized Hermite constants* usually denoted by $\gamma_{n,k}$ (see [Thu98]), namely they are related by

$$\bar{\alpha}_{n,k}^{2k} = \gamma_{n,k}. \quad (3)$$

The origin of this exponent $2k$ is the $1/k$ in the definition of \mathcal{V}_k , which we have imposed so that the functions α_k behave nicely with respect to homothety. This normalization has the additional advantage that the growth rate of the different $\bar{\alpha}_{n,k}$ (as a function of n) becomes the same for all k . Namely [Thu98, Cor. 2] and (3) show that $\log \bar{\alpha}_{n,k} = \frac{1}{2} \log n + O(1)$ (where the implicit constant depends on k).

2. It seems plausible that most lattices come close to realizing the Rankin constants, that is, for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} m(\{\Lambda \in \mathcal{L}_n : \forall k, \alpha_k(\Lambda) > \overline{\alpha}_{n,k} - \varepsilon\}) = 1.$$

Combined with the result of Thunder mentioned above, Proposition 3 may be viewed as supporting evidence for such a conjecture.

3. We take this opportunity to formulate an analogous question regarding the *covering radius*; that is, is it true that for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} m\left\{\Lambda \in \mathcal{L}_n : \text{covrad}(\Lambda) < \inf_{\Lambda' \in \mathcal{L}_n} \text{covrad}(\Lambda') + \varepsilon\right\} = 1,$$

where

$$\text{covrad}(\Lambda) = \inf\{r > 0 : \mathbb{R}^n = \Lambda + B(0, r)\}$$

and $B(0, r)$ is the Euclidean ball of radius r around the origin.

The proof of Propositions 2 and 3 relies on Thunder's work and on a variant of Siegel's formula [Sie45] which relates the Lebesgue measure on \mathbb{R}^n and the measure m on \mathcal{L}_n . We now review Siegel's method and Thunder's results.

In the sequel we consider $n \geq 2$ and $k \in \{1, \dots, n-1\}$ as fixed and omit, unless there is risk of confusion, the symbols n and k from the notation. Consider the (set valued) map $\Phi = \Phi_k^{(n)}$ that assigns to each lattice $\Lambda \in \mathcal{L}_n$ the following subset of the exterior power of $\bigwedge^k \mathbb{R}^n$:

$$\Phi(\Lambda) \stackrel{\text{def}}{=} \{\pm w_\Delta : \Delta \subset \Lambda \text{ a primitive subgroup with } r(\Delta) = k\},$$

where $w_\Delta \stackrel{\text{def}}{=} v_1 \wedge \dots \wedge v_k$ and $\{v_i\}_{i=1}^k$ is a basis for Δ (note that w_Δ is well-defined up to sign, and $\Phi(\Lambda)$ contains both possible choices). Let

$$\mathcal{V} = \mathcal{V}_k^{(n)} \stackrel{\text{def}}{=} \{v_1 \wedge \dots \wedge v_k : v_i \in \mathbb{R}^n\} \setminus \{0\}$$

be the variety of pure tensors in $\bigwedge^k \mathbb{R}^n$. For any compactly supported bounded Riemann integrable¹ function f on \mathcal{V} set

$$\hat{f} : \mathcal{L}_n \rightarrow \mathbb{R}, \quad \hat{f}(\Lambda) \stackrel{\text{def}}{=} \sum_{w \in \Phi(\Lambda)} f(w). \quad (4)$$

Then it is known (see [Wei82, Lemma 2.4.2]) that the (finite) sum (4) defines a function in $L^1(\mathcal{L}_n, m)$. This allows us to define a Radon measure $\theta = \theta_k^{(n)}$ on \mathcal{V} by the formula

$$\int_{\mathcal{V}} f d\theta \stackrel{\text{def}}{=} \int_{\mathcal{L}_n} \hat{f} dm, \text{ for } f \in C_c(\mathcal{V}). \quad (5)$$

Write $G = G_n \stackrel{\text{def}}{=} \text{SL}_n(\mathbb{R})$. There is a natural transitive action of G_n on \mathcal{V} and the stabilizer of $e_1 \wedge \dots \wedge e_k$ is the subgroup

$$H = H_k^{(n)} \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in G : A \in G_k, D \in G_{n-k} \right\}.$$

¹i.e. the measure of points at which f is not continuous is zero.

We therefore obtain an identification $\mathcal{V} \simeq G/H$ and view θ as a measure on G/H . It is well-known (see e.g. [Wei82]) that up to a proportionality constant there exists a unique G -invariant measure $m_{G/H}$ on G/H ; moreover, given Haar measures m_G, m_H on G and H respectively, there is a unique normalization of $m_{G/H}$ such that for any $f \in L^1(G, m_G)$

$$\int_G f dm_G = \int_{G/H} \int_H f(gh) dm_H(h) dm_{G/H}(gH). \quad (6)$$

We choose the Haar measure m_G so that it descends to our probability measure m on \mathcal{L}_n ; similarly, we choose the Haar measure m_H so that the periodic orbit $H\mathbb{Z}^n \subset \mathcal{L}_n$ has volume 1. These choices of Haar measures determine our measure $m_{G/H}$ unequivocally. It is clear from the defining formula (5) that θ is G -invariant and therefore the two measures $m_{G/H}, \theta$ are proportional. In fact (see [Sie45] for the case $k = 1$ and [Wei82, Lemma 2.4.2] for the general case),

$$m_{G/H} = \theta. \quad (7)$$

For $t > 0$, let $\chi = \chi_t : \mathcal{V} \rightarrow \mathbb{R}$ be the restriction to \mathcal{V} of the characteristic function of the ball of radius t around the origin, in $\bigwedge^k \mathbb{R}^n$, with respect to the natural inner product obtained from the Euclidean inner product on \mathbb{R}^n . Note that $\hat{\chi}(x) = 0$ if and only if $x \in \mathcal{S}_k^{(n)}(t^{1/k})$ and furthermore, $\hat{\chi}(x) \geq 1$ if $x \in \mathcal{L}_n \setminus \mathcal{S}_k^{(n)}(t^{1/k})$. It follows that

$$m(\mathcal{L}_n \setminus \mathcal{S}_k^{(n)}(t)) \leq \int_{\mathcal{L}_n} \widehat{(\chi_{t^k})} dm = \int_{\mathcal{V}} \chi_{t^k} d\theta. \quad (8)$$

Let V_j denote the volume of the Euclidean unit ball in \mathbb{R}^j and let ζ denote the Riemann zeta function. We will use an unconventional convention $\zeta(1) = 1$, which will make our formulae simpler. For $j \geq 1$, define

$$R(j) \stackrel{\text{def}}{=} \frac{j^2 V_j}{\zeta(j)} \quad \text{and} \quad B(n, k) \stackrel{\text{def}}{=} \frac{\prod_{j=1}^n R(j)}{\prod_{j=1}^k R(j) \prod_{j=1}^{n-k} R(j)}.$$

The following is [Thu98, Lemma 5]:

Theorem 4 (Thunder). *For $t > 0$, we have $\int_{\mathcal{V}} \chi_t dm_{G/H} = B(n, k) \frac{t^n}{n}$.*

(Note that in Thunder's notation, by [Thu98, §4], $c(n, k) = B(n, k)/n$.)

We will need to bound $B(n, k)$.

Lemma 5. *There is $C > 0$ so that for all large enough n and all $k = 1, \dots, n-1$,*

$$B(n, k) \leq \left(\frac{C}{n}\right)^{\frac{k(n-k)}{2}}. \quad (9)$$

Proof. In this proof c_0, c_1, \dots are constants independent of n, k, j . Because of the symmetry $B(n, k) = B(n, n - k)$ it is enough to prove (9) with $k \leq \frac{n}{2}$.

Using the formula $V_j = \frac{\pi^{j/2}}{\Gamma(\frac{j}{2}+1)}$ we obtain

$$\begin{aligned} B(n, k) &= \prod_{j=1}^k \frac{R(n-k+j)}{R(j)} = \prod_{j=1}^k \frac{\zeta(j)(n-k+j)^2 \frac{\pi^{(n-k+j)/2}}{\Gamma(\frac{n-k+j}{2}+1)}}{\zeta(n-k+j)j^2 \frac{\pi^{j/2}}{\Gamma(\frac{j}{2}+1)}} \\ &= \prod_{j=1}^k \frac{\zeta(j)}{\zeta(n-k+j)} \cdot \left(\frac{n-k+j}{j} \right)^2 \cdot \pi^{\frac{n-k}{2}} \cdot \frac{\Gamma(\frac{j}{2}+1)}{\Gamma(\frac{n-k+j}{2}+1)}. \end{aligned}$$

Note that $\zeta(s) \geq 1$ is a decreasing function of $s > 1$, so (recalling our convention $\zeta(1) = 1$) $\frac{\zeta(j)}{\zeta(n-k+j)} \leq c_0 \stackrel{\text{def}}{=} \zeta(2)$. It follows that for all large enough n and for any $1 \leq j \leq k$,

$$\frac{\zeta(j)}{\zeta(n-k+j)} \cdot \left(\frac{n-k+j}{j} \right)^2 \cdot \pi^{\frac{n-k}{2}} \leq c_0 n^2 \pi^{\frac{n-k}{2}} \leq 4^{\frac{n-k}{2}}. \quad (10)$$

According to Stirling's formula, there are positive constants c_1, c_2 such that for all $x \geq 2$,

$$c_1 \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e} \right)^x \leq \Gamma(x) \leq c_2 \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e} \right)^x.$$

We set $u \stackrel{\text{def}}{=} \frac{j}{2} + 1$ and $v \stackrel{\text{def}}{=} \frac{n-k}{2}$, so that $u + v \geq \frac{n-1}{4}$, and obtain

$$\begin{aligned} \frac{\Gamma(\frac{j}{2}+1)}{\Gamma(\frac{n-k+j}{2}+1)} &= \frac{\Gamma(u)}{\Gamma(u+v)} \leq \frac{c_2}{c_1} \sqrt{\frac{u+v}{u}} \frac{u^u}{(u+v)^{u+v}} \frac{e^{u+v}}{e^u} \\ &\leq c_3 e^v \frac{u^{u-1/2}}{(u+v)^{u+v-1/2}} = c_3 \left(\frac{e}{u+v} \right)^v \frac{1}{(1 + \frac{v}{u})^{u-1/2}}, \quad (11) \\ &\leq c_3 \left(\frac{4e}{n-1} \right)^{\frac{n-k}{2}}. \end{aligned}$$

Using (10) and (11) we obtain

$$B(n, k) \leq \left[c_3 4^{\frac{n-k}{2}} \left(\frac{4e}{n-1} \right)^{\frac{n-k}{2}} \right]^k = \left[c_3 \left(\frac{16e}{n-1} \right)^{\frac{n-k}{2}} \right]^k.$$

So taking $C > 16c_3e$ we obtain (9) for all large enough n . \square

Proof of Propositions 2 and 3. Let C be as in Lemma 5 and let $C_1 > C$. For Proposition 3, using (8), (7) and Theorem 4, for all sufficiently large n we have

$$\begin{aligned} m \left(\mathcal{L}_n \setminus \mathcal{S}_k^{(n)}(t_k) \right) &\leq B(n, k) \frac{t_k^{kn}}{n} \\ &\leq \frac{1}{n} \left(\frac{C}{n} \right)^{\frac{k(n-k)}{2}} \left(\frac{n}{C_1} \right)^{\frac{k(n-k)}{2}} = \frac{1}{n} \left(\frac{C}{C_1} \right)^{\frac{k(n-k)}{2}}. \end{aligned}$$

Multiplying by n and taking the maximum over k we obtain

$$n \max_{k=1,\dots,n} m\left(\mathcal{L}_n \setminus \mathcal{S}_k^{(n)}(t_k)\right) \leq \left(\frac{C}{C_1}\right)^{\frac{n-1}{2}} \rightarrow_{n \rightarrow \infty} 0.$$

The proof of Proposition 2 is identical using $t = 1$ instead of t_k . \square

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